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EMBEDDING RIEMANNIAN MANIFOLDS BY THEIR HEAT KERNEL

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Abstract

By embedding a class of closed Riemannian manifolds (satisfying some curvature assumptions and with diameter bounded from above) into the same Hilbert space, we interpret certain estimates on the heat kernel as giving a precompactness theorem on the class considered.

I. Introduction

The goal of this note is to interpret estimates (see [B1], [B2], [Bes2] and [G]) on the heat kernels of certain closed Riemannian manifolds as a precompactness theorem.

Define the set $\mathcal{M}_{n,k,D}$ of closed Riemannian manifolds by

$$\mathcal{M}_{n,k,D}=\big\{(M,g)\ \big|\ \dim M=n, \mathrm{Ricci}(g)\geq(n-1)kg \text{ and diameter } (M)\leq D\big\}.$$

As M. Gromov showed (see [GrLP, p. 65]), this space can be endowed with a metric structure with respect to which it is precompact. His technique consists in studying the manifolds from the point of view of metric spaces – a highly geometric point of view – and by approximating them by a finite number of points.

In our set up, we exhibit an embedding of *any* Riemannian manifold belonging to $\mathcal{M}_{n,k,D}$ into the space ℓ^2 of real valued, square integrable series. This embedding is built using the heat kernel of the manifolds. The curvature and diameter assumptions, when translated on the heat kernel, lead to the fact that the image of $\mathcal{M}_{n,k,D}$ by the embedding is bounded in a subspace h^1 which in turn embeds compactly into ℓ^2. Pulling back the

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Hausdorff distance between subsets of ℓ^2 we obtain distances on $\mathcal{M}_{n,k,D}$ for which this space is precompact.

Our point of view is more related to the spectral aspect and as such is less geometric. However, among the results that are spectral (or analytic) translations of geometric hypothesis very few have a geometric feedback; this is one example.

A preliminary version of this work appeared in 1986 in preprint form, with some minor incomplete arguments. The present article contains several improvements. In an unpublished paper H. Muto ([Mu]) made some remarks on our 1986 preprint. Very recently, A. Kasue and H. Kumura ([KKu]), using the same basic idea, described a new topology on the set of Riemannian metrics on a given closed Riemannian manifold and proved another precompactness theorem.

In section IV, we define distances between two Riemannian manifolds, which we call *spectral distances*, giving a meaning to the notion of convergence of Riemannian manifolds in the sense of the spectrum. This notion would have to be compared with the one used in K. Fukaya's work (see [Fu1], [Fu2], [Fu3] and [Fu4]).

The necessary estimates on the heat kernel (Theorem 3) are recalled in section II; in sections III and IV, we define the embedding and the spectral distances (Definition 7 and Theorem 10). Section V deals with the precompactness theorem (Theorem 14). Spectral distances are compared to the Lipschitz distance in section VI (Theorem 17); this section also contains a result on C^0-approximation of eigenfunctions (Theorem 21) which is of independent interest. In section VII, we show that the spectrum is continuous with respect to the spectral distances. Section VIII contains further comments.

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II. Estimating the Heat Kernel

Let (M,g) be a closed (*i.e.* compact, without boundary) Riemannian manifold, and Δ its Laplace–Beltrami operator.

The heat semigroup is the family of self adjoint operators, $e^{-t\Delta}$ $(t>0)$. It has a smooth kernel. More precisely

$$\forall f \in L^2(M)\ , \quad (e^{-t\Delta}f)(x) = \int_M k_M(t;x,y)f(y)dy$$

where k_M is a smooth function of (t,x,y) ($t>0$, x and y in M). It is the heat kernel of the Riemannian manifold M. For a review of the main properties of k_M see [B1], [BeGaM] and [C].

The manifold being closed, the spectrum of the Laplacian is a sequence of eigenvalues $\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \cdots \nearrow +\infty$. Given an $L^2(M)$–orthonormal basis $\{\varphi_i\}_{i=0}^{\infty}$ of real eigenfunctions of the Laplacian, one can write

$$k(t;x,y) = \sum_{i=0}^{\infty} e^{-\lambda_i t}\varphi_i(x)\varphi_i(y) .$$

The operator $e^{-t\Delta}$ is trace class and

$$Z_M(t) = \operatorname{Trace}(e^{-t\Delta}) = \int_M k_M(t;x,x)dx = \sum_{i=0}^{\infty} e^{-\lambda_i t} .$$

Let us recall the following

THEOREM 1 ([BG], [BBesG]). *Let (M,g) be an n–dimensional closed Riemannian manifold and define*

$$r_{\min}(M) = \inf\{\operatorname{Ricci}(u,u) \mid u \textit{ unit tangent vector to } M\}$$

$$d = d(M) = \textit{ diameter of } M .$$

If (M,g) satisfies $r_{\min}(M)d(M)^2 \geq (n-1)\varepsilon\alpha^2$ for some $\varepsilon \in \{-1,0,1\}$ and some positive number α, then

$$\operatorname{Vol}(M)k_M(t;x,x) \leq \operatorname{Vol}\left(S^n(R)\right)k_{S^n(R)}(t;p,p) = Z_{S^n(R)}(t) = Z_{S^n(1)}(t/R^2),$$

where p is any point in the sphere $S^n(R)$ of radius $R = a(n,\varepsilon,\alpha)d(M)$ and where $a(n,\varepsilon,\alpha)$ is a real number which is given explicitly in terms of n,ε,α.

Remarks 2: i) Theorem 1 is the analytic translation of the generalized Paul Lévy–Gromov isoperimetric inequality given in [BBesG]. Any other sharp isoperimetric inequality would lead to a similar result.

ii) The last equality comes from the invariance of $k_{S^n(R)}(t;p,p)$ on the two-point homogeneous manifold $S^n(R)$.

iii) The proof of Theorem 1 uses the so-called symmetrization process ([Bes2]). Given a positive function u on M, let $S(u)$ denote the function obtained on $S^n(R)$ by symmetrizing u around a given point $p \in S^n(R)$. If u is not positive, define $S(u)$ to be $S(|u|)$. One can then prove the following inequality: for any functions u, v on M,

$$\langle e^{-t\Delta_M}(u), v\rangle \leq \langle e^{-t\Delta_{S^n(R)}}(S(u)), S(v)\rangle$$

where $\langle \cdot , \cdot \rangle$ denotes the L^2 inner product (either on M or on $S^n(R)$). This inequality implies Theorem 1. We will need an improved version of Theorem 1.

Let us recall that $\mathcal{M}_{n,k,D}$ is the set of closed Riemannian manifolds of dimension n, satisfying:

(a) $r_{\min}(M) \geq (n-1)k$;

(b) diameter of $M = d(M) \leq D$;

where k is a real number (without any sign assumption) and D a positive real number.

We then have

THEOREM 3. *With the above notations, there exist positive constants $A(n,k,D)$, $B(n,k,D)$ and $C(n,k,D)$, depending only on n, k and D, such that, for any $(M,g) \in \mathcal{M}_{n,k,D}$:*

i) *j–th eigenvalue of $M = \lambda_j(M) \geq A(n,k,D)j^{2/n}$,*

ii) *$N_M(\lambda) = \#\{j | \lambda_j(M) \leq \lambda\} \leq 1 + B(n,k,D)\lambda^{n/2}$,*

iii) *for all $x \in M$ and $\alpha \geq 0$,*

$$\sum_{j\geq 1} \lambda_j^{\alpha}(M) \exp\big(-t\lambda_j(M)\big)\varphi_j^2(x) \leq \frac{C(n,k,D)}{\mathrm{Vol}(M)}(\alpha+1)t^{-(n+2\alpha)/2} .$$

Proof: 1) If $k \geq 0$, $r_{\min}(M)d^2(M) \geq 0$ and if $k < 0$, $r_{\min}(M)d^2(M) \geq (n-1)kD^2$; we can then apply Theorem 1, with ε and α depending only on k and D.

2) As a consequence,

$$\begin{aligned} Z_M(t) &= \int_M k_M(t;x,x)dx \leq \mathrm{Vol}(M) \sup_x \{k_M(t;x,x)\} \\ &\leq \mathrm{Vol}\,(S^n(R))k_{S^n(R)}(t;p,p) = Z_{S^n(R)}(t) = Z_{S^n(1)}(t/R^2) . \end{aligned}$$

The trace of the heat operator is thus uniformly bounded on the set $\mathcal{M}_{n,k,D}$.

The first term in the summation for $Z_M(t)$ and $Z_{S^n(R)}(t)$ being 1, the above inequality is equivalent to $(Z_M(t)-1) \leq (Z_{S^n(R)}(t)-1)$.

Furthermore the asymptotic expansion of $Z_{S^n(1)}(t)$ as t goes to zero shows that there exists a constant $b(n)$ such that, for any positive t,

$$\big(Z_{S^n}(t) - 1\big) \leq b(n)t^{-n/2} ,$$

where S^n is the standard unit sphere.

3) As a consequence

$$j \le N(\lambda_j) - 1 \le e \sum_{0<\lambda_i\le\lambda_j} e^{-\lambda_i/\lambda_j} \le e(Z_M(1/\lambda_j)-1) \le e(Z_{S^n}(1/\lambda_j R^2)-1)$$

$$j \le N(\lambda_j) - 1 \le \left(eb(n)R^n\right)\lambda_j^{n/2}$$

recall that $R = a(n,\varepsilon,\alpha)d(M) \le a(n,\varepsilon,\alpha)D$ and that $a(n,\varepsilon,\alpha)$ only depends on n, k, D. This proves i) and ii).

4) For $x \in M$, let us define the positive measure μ_x on $\mathbf{R}_+$ by

$$d\mu_x = \sum_{j\ge1} \varphi_j^2(x)\delta_{\lambda_j} \quad \text{(we avoid } \lambda_0 = 0\text{)}$$

where δ_{λ_j} is the Dirac measure at $\lambda_j \in \mathbf{R}$. Then, omitting the mention to M for the sake of simplicity,

$$\sum_{j\ge1} \lambda_j^\alpha e^{-t\lambda_j}\varphi_j^2(x) = \int_{\mathbf{R}} \lambda^\alpha e^{-t\lambda} d\mu_x(\lambda)$$

integrating by parts gives

$$0 \le \int_{\mathbf{R}} \lambda^\alpha e^{-t\lambda} d\mu_x(\lambda) \le \int_{\mathbf{R}} \lambda^{\alpha-1}(\lambda t + \alpha)e^{-t\lambda}\mu_x([0,\lambda])d\lambda$$

with

$$\mu_x([0,\lambda]) = \sum_{0<\lambda_j\le\lambda} \varphi_j^2(x)\ .$$

As previously, Theorem 1 implies that

$$\sum_{0<\lambda_j\le\lambda} \varphi_j^2(x) \le e \sum_{0<\lambda_j\le\lambda} e^{-\lambda_j/\lambda}\varphi_j^2(x) \le e \sup_{x\in M}\left(k_M\left(\frac{1}{\lambda}; x, x\right) - \frac{1}{\mathrm{Vol}(M)}\right)$$
$$\le \frac{e}{\mathrm{Vol}(M)}\left(Z_{S^n}\left(\frac{1}{\lambda R^2}\right) - 1\right) \le E(n,k,D)\frac{\lambda^{n/2}}{\mathrm{Vol}(M)} \quad (\lambda \ge 0)$$

and hence

$$\int_{\mathbf{R}} \lambda^{\alpha-1}(\lambda t+\alpha)e^{-t\lambda}\mu_x([0,\lambda])d\lambda \le \frac{E(n,k,D)}{\mathrm{Vol}(M)}\int_0^\infty \lambda^{\alpha-1}(\lambda t+\alpha)e^{-t\lambda}\lambda^{n/2}d\lambda$$

and iii) follows. □

III. Embedding a Riemannian Manifold into a Hilbert Space

Our purpose here is to "uniformly" embed the elements of $\mathcal{M}_{n,k,D}$ into the same Hilbert space; a natural candidate is the space ℓ^2.

Let ℓ^2 be the Hilbert space of real sequences $\{a_i\}_{i\geq 1}$ such that $\sum |a_i|^2 < \infty$. Let h^1 be the Hilbert space of real sequences $\{a_i\}_{i\geq 1}$ such that $\sum |a_i|^2(1+i^{2/n}) < \infty$ (n is the dimension of the manifolds belonging to $\mathcal{M}_{n,k,D}$), with its natural norm. Rellich's Theorem asserts that the embedding $h^1 \hookrightarrow \ell^2$ is compact.

DEFINITION 4. *Given an n-dimensional closed Riemannian manifold M and an orthonormal basis a of real eigenfunctions of the Laplacian of M, one defines the family of maps*

$$\psi_t^a : M \to \ell^2 \quad \textit{for} \quad t > 0,$$

$$x \to \sqrt{2}(4\pi)^{n/4} t^{(n+2)/4} \left\{e^{-\lambda_j t/2} \varphi_j^a(x)\right\}_{j\geq 1}$$

(notice that we have suppressed the constant eigenfunction for convenience).

THEOREM 5. *Fix an n-dimensional closed Riemannian manifold (M, g) and an orthonormal basis a of real eigenfunctions of its Laplacian. Let* can *denote the Euclidean scalar product on ℓ^2.*

i) *For all positive t, the map ψ_t^a is an embedding of M into ℓ^2;*

ii) *The pulled-back metric $(\psi_t^a)^*$ can is asymptotic to the metric g of M when t goes to zero. More precisely, $(\psi_t^a)^* \mathrm{can} = g + {t/3}({1/2}\,\mathrm{Scal}_g \cdot g - \mathrm{Ric}_g) + O(t^2)$ when $t \to 0_+$ (Scal_g is the scalar curvature and Ric_g the Ricci curvature tensor of the metric g).*

Remarks 6: i) The embedding used in [Gr] is defined by the distance function and is an isometry for the metric structure. It is then geometric, but the target space is a Banach space. Our embedding is more analytic and the target space is a Hilbert space.

ii) Up to a constant factor the embedding ψ_t may also be viewed as the composition of the map $x \mapsto k_M(t/2; x, \cdot)$ from M into $L^2(M, v_g)$ with the linear isometry from $L^2(M, v_g)$ onto ℓ^2 given by the choice of an orthonormal basis a of eigenfunctions.

iii) When (M, g) is an irreducible homogeneous space, the pulled-back metric $(\psi_t^a)^*$ can is homothetic to the metric g for any t. This comes from the fact that k_M (and hence $(\psi_t^a)^*$ can) is invariant by isometries.

Proof: Recall that $a = \{\varphi_j\}_{j=0}^{\infty}$ is a fixed orthonormal basis of real eigenfunctions of the Laplacian.

1) The map $\Phi_t^a : x \to \{e^{-\lambda_j t/2}\varphi_j(x)\}_{j\geq 1}$ is continuous in (t,x) for $t>0$ and x in M; indeed if (t_n,x_n) converges to (t,x)

$$\left\|\Phi_{t_n}^a(x_n) - \Phi_t^a(x)\right\|_{\ell^2}^2 = \sum_{j\geq 1}\left|e^{-\lambda_j t_n/2}\varphi_j(x_n) - e^{-\lambda_j t/2}\varphi_j(x)\right|^2$$

$$= k_M(t_n; x_n, x_n) + k_M(t; x, x) - 2k_M\left(\frac{t_n+t}{2}; x_n, x\right)$$

which goes to 0 as n goes to $+\infty$. The continuity of ψ_t^a follows. Thus $\psi_t^a(M)$ is a compact subset of ℓ^2.

2) The map ψ_t^a is also one to one, for all $t>0$:

$$\psi_t^a(x) = \psi_t^a(y) \Longleftrightarrow \varphi_j(x) = \varphi_j(y) \text{ for all } j \geq 0$$

but $\{\varphi_j\}_{j\geq 0}$ is a basis of $L^2(M)$ and hence separates the points, thus $x=y$.

The map is then continuous and one–to–one from M which is compact onto $\psi_t^a(M)$, so it is a homeomorphism. Let us now suppose that $d\psi_t^a(X) = 0$ for some $X \in T_xM$. It follows that $d\varphi_j(X) = 0$ for every j and that $du(X) = 0$ for any smooth function u, which implies $X = 0$.

3) If V is a tangent vector to M at x, then

$$d\Phi_t^a(V) = \left\{e^{-\lambda_j t/2}d_x\varphi_j(V)\right\}_{j\geq 1}\ .$$

The Riemannian metric defined on M by pulling back the metric *can* (the Euclidean metric of ℓ^2) is

$$\langle V,V\rangle_t = \left\|d\Phi_t^a(V)\right\|_{\ell^2}^2 = \sum_{j\geq 1} e^{-\lambda_j t}\left|d_x\varphi_j(V)\right|^2 = (d_S k)_{(x,x)}(V,V)\ .$$

Here, for a function f defined on $M\times M$, $d_S f$ is the "mixed second derivative" of f; more precisely, if d_1 (resp. d_2) is the differential with respect to the first space variable (resp. the second)

$$d_S f_{(x,x)} = d_{2,y}d_{1,x}f(x,y)\big|_{(x,x)} \quad \text{(i.e. taken on the diagonal).}$$

Let us recall the Minakshisundaram–Pleijel asymptotic expansion (see [BeGaM, p. 204]): there exist C^∞ functions u_i on $M\times M$ such that for any integer p and for all (x,y) in $M\times M$ close to the diagonal

$$k(t;x,y) = \frac{1}{(4\pi t)^{n/2}}e^{-r^2(x,y)/4t}\left(u_0(x,y) + tu_1(x,y) + \cdots + t^p u_p(x,y) + O(t^{p+1})\right)$$

where $r(x,y)$ = Riemannian distance between x and y in M (assumed to be small). It follows from the proof that this expansion can be differentiated term by term as many times as needed (see [BeGaM, p. 213] or [C, p. 154]).

Let us define

$$U(t;x,y) = u_0(x,y) + t\ u_1(x,y) + \cdots + t^p\ u_p(x,y) + O(t^{p+1})\ .$$

Differentiating then gives

$$\begin{aligned} d_1 k &= \frac{1}{(4\pi t)^{n/2}} e^{-r^2/4t} \left[-\frac{d_1(r^2)}{4t} U + d_1 U \right] \\ d_S k &= \frac{1}{(4\pi t)^{n/2}} \left[-\frac{d_S(r^2)}{4t} U - \frac{d_1(r^2)}{4t} d_2 U + d_S U \right. \\ &\qquad \left. - \left(\frac{d_2(r^2)}{4t} \right) \left(-\frac{d_1(r^2)}{4t} U + d_1 U \right) \right] e^{-r^2/4t}\ . \end{aligned}$$

The function $r^2(x,y)$ is smooth on $M \times M$, provided that x and y are close enough. Then by taking normal coordinates in M, one can easily see that,

$$d_1(r^2)(V) = d_2(r^2)(V) = 0 \quad \text{at} \quad (x,x)\ .$$

Finally,

$$(d_S k)_{(x,x)}(V,V) = \frac{1}{(4\pi t)^{n/2}} \left[-\frac{d_S(r^2)_{(x,x)}(V,V)}{4t} U(t;x,x) + (d_S U)_{(x,x)}(V,V) \right]\ .$$

We then have the

LEMMA. *With the above notation, for any $V \in T_x M$,*
i) $d_S(r^2)_{(x,x)}(V,V) = -2 g_x(V,V)$ *(g is the metric of M)*
ii) $(d_S u_0)_{(x,x)}(V,V) = -{}^1\!/_{3!}\,\mathrm{Ricci}_x(V,V)$.

Proof of the lemma: 1) Let x_t be the geodesic segment $x_t = \exp_x(tV)$. The derivative d_S depending only on the value of the vector field at the point under consideration we have

$$(d_S r^2)_{(x,x)}(V,V) = \frac{\partial}{\partial t} \frac{\partial}{\partial s}_{|t=s=0} r^2(x_t, x_s)\ ,$$

but $r^2(x_t, x_s) = |t-s|^2 \|V\|^2$. This proves i).

2) Let us recall that

$$u_0(x,y) = \left[\theta(x,y)\right]^{-1/2} \text{(for } x \text{ and } y \text{ close enough)}$$

where

$$\theta(x,y) = \frac{\text{volume density at } y \text{ read in the normal coordinates around } x}{r^{n-1}}$$

with $r = r(x,y)$ (see [BeGaM], p. 208).

For the sake of simplicity let us assume that $|V| = 1$, the general case being obtained by an appropriate scaling.

The Taylor expansion of $\theta(x,y)$ about x ([BeGaM, p. 100]) is a consequence of the second theorem of E. Cartan. This gives that the Taylor expansion of $\theta(s,t) = \theta(x_s, x_t)$ about (x,x) is

$$\theta(s,t) = 1 - \mathrm{Ricci}\,\big(\dot{x}(s), \dot{x}(s)\big)\frac{|t-s|^2}{3!} + O\big(|t-s|^3\big)$$

which by an easy computation gives

$$d_S u_0(V,V) = \frac{\partial^2(\theta^{-1/2})}{\partial t \partial s}(0,0) = -\frac{1}{3!}\,\mathrm{Ricci}_x(V,V)\ .$$

Finally,

$$d_S k_{(x,x)}(V,V) = \frac{1}{(4\pi t)^{n/2}}\left\{\frac{1}{2t} g(V,V)\big[u_0(x,x) + t u_1(x,x) + O(t^2)\big]\right.$$
$$\left. -\frac{1}{3!}\,\mathrm{Ricci}_{(x,x)}(V,V) + O(t)\right\}\ .$$

Let us recall that

$$u_0(x,x) = \theta(x,x)^{-1/2} = 1$$

$$u_1(x,x) = \frac{\tau(x)}{6} \quad \text{where } \tau \text{ is the scalar curvature of } M\ .$$

$$d_S k_{(x,x)}(V,V) = \frac{1}{2(4\pi)^{n/2} t^{\frac{n+2}{2}}}\left[g(V,V) + \frac{t}{3}\left(\frac{\tau}{2} g(V,V) - \mathrm{Ricci}(V,V)\right) + O(t^2)\right]$$

$$d_S k = \frac{1}{2(4\pi)^{n/2} t^{\frac{n+2}{2}}}\left[g + \frac{t}{3}\left(\frac{\tau}{2} g - \mathrm{Ricci}\right) + O(t^2)\right]$$

and

$$(\psi_t^a)^* = g + \frac{t}{3}\left(\frac{\tau}{2} g - \mathrm{Ricci}\right) + O(t^2)$$

which proves the theorem. □

IV. Spectral Distances

The embedding ψ_t^a of section III (Definition 4) depends on the choice of an orthonormal basis a of eigenfunctions (due for example to multiple eigenvalues). Given a Riemannian manifold (M,g), we decompose the space $L^2(M,g)$ as $L^2(M,g) = \bigoplus_{i=0}^{\infty} E_i$, where the E_i's are the eigenspaces of the Laplacian Δ_g (corresponding to increasing eigenvalues μ_i). We let $\mathcal{B}(M,g) = \prod_{i=1}^{\infty} \mathcal{B}(E_i)$ denote the set of corresponding orthonormal bases. The space $\mathcal{B}(E_i)$ can be identified with $O(\dim E_i)$ and hence $\mathcal{B}(M,g)$ is a compact set with respect to the product topology. This topology can be described by the following distance.

Given a Euclidean space E, the distance $d_E(a,b)$ between two orthonormal bases $a,b \in \mathcal{B}(E)$ is the usual Euclidean distance between the identity matrix and the transition matrix from a to b. This distance satisfies $d_E(a,b) \leq 2\sqrt{\dim E}$ for all $a,b \in \mathcal{B}(E)$. The distance d on $\mathcal{B}(M,g)$ is defined by

$$d(a,b)^2 = \sum_{i=1}^{\infty} \mu_i^{-N} d_{E_i}\big(a|E_i, b|E_i\big)^2$$

where the series in the right hand side converges if we choose $N > n/2$ (as the dimension n of the manifolds under consideration is fixed we may fix N once and for all).

The fact that the series converges follows from the bounds on multiplicities given by Theorem 3. We now define a slightly different embedding (which differs from ψ_t^a by a factor).

DEFINITION 7. *Given an n-dimensional closed Riemannian manifold M and an orthonormal basis $a = \{\varphi_j^a\}$ of eigenfunctions of the Laplacian of M, one defines a family of maps $I_t^a : M \to \ell^2$ by*

$$I_t^a(x) = \sqrt{\operatorname{Vol}(M)}\{e^{-\lambda_j t/2}\varphi_j^a(x)\}_{j\geq 1} .$$

Notice that the family $\{I_t^a\}_t$ is globally invariant under scaling of the metric.

THEOREM 8. *Given an n-dimensional closed Riemannian manifold (M,g), the map $I : \mathbf{R}_+^* \times \mathcal{B}(M,g) \times M \longrightarrow \ell^2$, defined by $I(t,a,x) = I_t^a(x)$, is continuous. More precisely, we have*

$$\begin{aligned}\|I_t^a(x) - I_s^b(y)\|_{\ell^2}^2 \leq \operatorname{Vol}(M)\Big\{ & k(t;x,x) + k(s;y,y) - 2k\left(\frac{t+s}{2};x,y\right) \\ & + d(a,b)\big(k_N(t;x,x)k_N(s;y,y)\big)^{\frac{1}{2}}\Big\}\end{aligned} \qquad (*)$$

where $k^{(N)}(t,x,y) = \sum_{j\geq 1} \lambda_j^{N/2} e^{-\lambda_j t} \varphi_j^a(x)^2$.

Proof: Take $t, s \in \mathbf{R}_+^\bullet$, $a, b \in \mathcal{B}(M,g)$, and $x, y \in M$. One can write:

$$\left\|I_t^a(x) - I_s^b(y)\right\|_{\ell^2}^2 = \operatorname{Vol}(M)\left\{k(t;x,x) + k(s;y,y) - 2k\left(\frac{t+s}{2};x,y\right)\right\}$$
$$-2\operatorname{Vol}(M)\sum_{j\geq 1} e^{-\lambda_j(t+s)/2}\left[\varphi_j^a(x)\varphi_j^b(y) - \varphi_j^a(x)\varphi_j^a(y)\right].$$

Call the last summation A. In each eigenspace one can write

$$\varphi_j^b(x) = \sum_k \alpha_{kj}(b,a)\varphi_k^a(x)$$

where $\alpha_{kj}(b,a)$ is the transition matrix (which does not depend on x). Denote by $\{\mu_j\}_{j=0}^{\infty}$ the eigenvalues (as points of the spectrum) and by $m(\mu_j)$ the multiplicity of μ_j. One can write A as

$$A = \sum_{i\geq 1} e^{-\mu_i(t+s)/2} \sum_{j,k=1}^{m(\mu_i)} \varphi_j^a(x)\varphi_k^a(y)\big(\alpha_{kj}(b,a) - \delta_{kj}\big)$$

$$|A| \leq \sum_{i\geq 1} e^{-\mu_i(t+s)/2}\left(\sum_{j=1}^{m(\mu_i)} \varphi_j^a(x)^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{m(\mu_i)} \varphi_k^a(y)^2\right)^{\frac{1}{2}} d_{E_{\mu_i}}\left(a|E_{\mu_i}, b|E_{\mu_i}\right)$$

$$|A| \leq d(a,b)\sum_{i\geq 1} \mu_i^{N/2} e^{-\mu_i(t+s)/2}\left(\sum_{j=1}^{m(\mu_i)} \varphi_j^a(x)^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{m(\mu_i)} \varphi_k^a(y)^2\right)^{\frac{1}{2}}$$

$$|A| \leq d(a,b)\left(\sum_{j\geq 1} \lambda_j^{N/2} e^{-\lambda_j t}\varphi_j^a(x)^2\right)^{\frac{1}{2}}\left(\sum_{j\geq 1} \lambda_j^{N/2} e^{-\lambda_j s}\varphi_j^a(y)^2\right)^{\frac{1}{2}}.$$

Recall that $k^{(N)}(t,x,x) = \sum_{j\geq 1} \lambda_j^{N/2} e^{-\lambda_j t}\varphi_j^a(x)^2$ (this function is controlled by Theorem 3 (iii)). It follows from the above computations that

$$\left\|I_t^a(x) - I_s^b(y)\right\|_{\ell^2}^2 \leq \operatorname{Vol}(M)\left\{k(t;x,x) + k(s;y,y) - 2k\left(\frac{t+s}{2};x,y\right)\right.$$
$$\left. + d(a,b)\big(k^{(N)}(t;x,x)k^{(N)}(s;y,y)\big)^{\frac{1}{2}}\right\}. \qquad (*)$$

When $(t,a,x) \to (s,b,y)$, the right–hand side goes to zero. □

Let HD denote the Hausdorff distance between compact subsets of ℓ^2. Given two Riemannian manifolds M and M', we define

$$\begin{aligned} d_t(M,M') = \max\Big\{ & \sup_{a\in\mathcal{B}(M)} \inf_{a'\in\mathcal{B}(M')} HD\big(I_t^a(M), I_t^{a'}(M')\big), \\ & \sup_{a'\in\mathcal{B}(M')} \inf_{a\in\mathcal{B}(M)} HD\big(I_t^{a'}(M'), I_t^a(M)\big)\Big\}. \end{aligned} \tag{9}$$

THEOREM 10. *For any fixed $t > 0$, d_t is a* distance *between isometry classes of Riemannian manifolds. In particular, $d_t(M,M') = 0$ if and only if the Riemannian manifolds M and M' are isometric.*

We will call this family of distances *spectral distances.*

Proof: The triangle inequality is easy (same proof as for the Hausdorff distance). Assume that $d_t(M,M') = 0$. For every $a' \in \mathcal{B}(M')$, there exists a sequence $(a_n)_{n\in\mathbf{N}}$ in $\mathcal{B}(M)$ such that the limit of $HD(I_t^{a_n}(M), I_t^{a'}(M'))$ is zero. By compactness of $\mathcal{B}(M)$, a subsequence converges to some element a of $\mathcal{B}(M)$. It follows from the triangle inequality and from inequality $(*)$ in Theorem 8 that $HD\big(I_t^a(M), I_t^{a'}(M')\big) = 0$. Let $a = \{\varphi_i\}_{i\geq 1}$ and $a' = \{\varphi'_i\}_{i\geq 1}$. From the definition of HD and the compactness of the images, one deduces

$$\begin{aligned} &\forall x \in M \quad \exists y'_t \in M' \text{ s.t. for } i \geq 1, \\ &\sqrt{\mathrm{Vol}(M)}e^{-\lambda_i t/2}\varphi_i(x) = \sqrt{\mathrm{Vol}(M')}e^{-\lambda'_i t/2}\varphi'_i(y'_t); \end{aligned} \tag{11i}$$

$$\begin{aligned} &\forall y' \in M' \quad \exists x_t \in M \text{ s.t. for } i \geq 1, \\ &\sqrt{\mathrm{Vol}(M)}e^{-\lambda_i t/2}\varphi_i(x_t) = \sqrt{\mathrm{Vol}(M')}e^{-\lambda'_i t/2}\varphi'_i(y'). \end{aligned} \tag{11ii}$$

Step 1. Because eigenfunctions separate the points in the manifold, the point y'_t (resp. x_t) is uniquely defined and hence the correspondence $x \to y'_t$ (resp. $y' \to x_t$) is a well-defined map f_t (resp. h_t). Furthermore, $f_t \circ h_t = Id_{M'}$ and $h_t \circ f_t = Id_M$. It is easy to see that f_t and h_t are continuous.

Step 2. The maps f_t and h_t are C^∞ diffeomorphisms.

LEMMA 12. *For any $x_0 \in M$, there exist $n = \dim(M)$ eigenfunctions $\varphi_{i_1}, \dots, \varphi_{i_n}$ such that the gradient vectors $\nabla\varphi_{i_k}(x_0)$ span $T_{x_0}M$.*

Proof of the Lemma: If not, there exists a proper subspace V of $T_{x_0}M$ which contains any finite linear combination of the $\nabla\varphi_i(x_0)$. Any smooth real function u can be written as $u = a_0 + \sum a_i\varphi_i$ (for $a_i \in \mathbf{R}$) where the series converges in the C^1 topology. It follows that $\nabla u(x_0) = \sum a_i \nabla\varphi_i(x_0)$. Now, any vector in $T_{x_0}M$ can be written as $\nabla u(x_0)$ for some function u : this leads to a contradiction.

Proof of Step 2: Take x_0 and $\varphi_{i_1}, \dots, \varphi_{i_n}$ as indicated in Lemma 12 and define a map, $F : M \times M' \to \mathbf{R}^n$, by $F(x,y') = \big(\varphi_{i_k}(x) - c_{i_k}(t)\varphi'_{i_k}(y')\big)_{k=1}^{n}$ where $c_i(t) = e^{(\lambda_i - \lambda'_i)t/2}\big(\mathrm{Vol}(M')/\mathrm{Vol}(M)\big)^{1/2}$. Take $y'_0 = f_t(x_0)$ (and hence, $h_t(y'_0) = x_0$). The map h_t satisfies $F(h_t(y'), y') = 0$. From Lemma 12, it follows that the partial differential of F w.r.t the first variable at (x_0, y'_0) is an isomorphism and hence h_t is locally smooth at y'_0. It follows that h_t is smooth. The same proof shows that f_t is smooth too. □

Step 3. The manifolds M and M' are isometric.

The direct image $(f_t)_*\mathrm{dvol}_M$ of the Riemannian measure can be written as $a_t\,\mathrm{dvol}_{M'}$ (where a_t is the Jacobian of h_t). Integrating relation (11i), we obtain, for $i \geq 1$:

$$
\begin{aligned}
0 &= \sqrt{\mathrm{Vol}(M)}e^{-\lambda_i(t/2)} \int_M \varphi_i(x)\mathrm{dvol}_M(x) \\
&= \sqrt{\mathrm{Vol}(M')}e^{-\lambda'_i(t/2)} \int_M \varphi'_i \circ f_t(x)\mathrm{dvol}_M(x) \\
&= \sqrt{\mathrm{Vol}(M')}e^{-\lambda'_i(t/2)} \int_{M'} \varphi'_i(y)a_t(y)\ \mathrm{dvol}_{M'}(y)\ .
\end{aligned}
$$

It follows that a_t is orthogonal to the φ'_i, $i \geq 1$ and hence that a_t is a constant. Because a_t is a constant, $\int_{M'}(f_t)_*\mathrm{dvol}_M = a_t\,\mathrm{Vol}(M') = \int_M \mathrm{dvol}_M = \mathrm{Vol}(M)$ and hence $a_t = \mathrm{Vol}(M)/\mathrm{Vol}(M')$. Integrating relation (11i) squared gives $\mathrm{Vol}(M)e^{-\lambda_i t} = \mathrm{Vol}(M')a_t e^{-\lambda'_i t}$; it follows that $\lambda_i = \lambda'_i$, for $i \geq 1$ and hence $\mathrm{Vol}(M) = \mathrm{Vol}(M')$.

Using relation (11i) again and the fact that $\lambda_i = \lambda'_i$, for $i \geq 1$, we conclude that $\Delta' \circ f_t^* = f_t^* \circ \Delta$, i.e. the diffeomorphism f_t intertwines the Laplacians. Taking the principal symbols, this implies that f_t is an isometry. □

This concludes the proof of Theorem 10.

Remark 13: The above proof applies to kernels of the form $\sum f(\lambda_j)\varphi_j(x)\varphi_j(y)$ with f injective and decreasing sufficiently fast at infinity. The knowledge of the kernel determines the eigenfunctions and the eigenvalues and hence the Riemannian manifold up to isometry.

V. A Precompactness Theorem

THEOREM 14. *For any $t > 0$, the space $\mathcal{M}_{n,k,D}$ is d_t-precompact.*

Proof: Let $h^1 \subset \ell^2$ be the space of sequences $\{a_j\}$ such that $\sum\limits_{j\geq 1}(1+j^{2/n})a_j^2 < \infty$. By Rellich's theorem, the inclusion $h^1 \hookrightarrow \ell^2$ is compact. For $t > 0$ and for $a = \{\varphi_j\} \in \mathcal{B}(\mathcal{M})$, we have

$$\begin{aligned}\left\|I_t^a(x)\right\|_{h^1}^2 &= \operatorname{Vol}(M)\sum_{j\geq 1}(1+j^{2/n})e^{-\lambda_j t}\varphi_j^2(x)\\ &\leq E(n,k,D)\operatorname{Vol}(M)\sum_{j\geq 1}(1+\lambda_j)e^{-\lambda_j t}\varphi_j^2(x)\\ &\leq F(n,k,D)t^{-n/2}(1+t^{-1})\end{aligned}$$

where $E(n,k,D)$ and $F(n,k,D)$ are universal constants; the first inequality follows from Theorem 3(i), the second from Theorem 3(iii). As a consequence, the set $\{I_t^a(x) | x \in M \in \mathcal{M}_{n,k,D}, a \in \mathcal{B}(M)\}$ is a bounded subset in h^1 and hence relatively compact in ℓ^2: let $\mathcal{K}$ denote its closure in ℓ^2. We now consider the compact sets $I_t^a(M) \subset \mathcal{K}$ for $M \in \mathcal{M}_{n,k,D}$ and $a \in \mathcal{B}(M)$ (recall that t is fixed).

Recall the following easy lemma:

LEMMA 15. *Let (E,δ) be a metric space. Let $\mathcal{F}(E)$ denote the set of non-empty closed subsets of E, equipped with the Hausdorff distance h_δ associated with δ. If the metric space (E,δ) is precompact, so is the metric space $(\mathcal{F}(E), h_\delta)$.*

Consider $\mathcal{F}(\mathcal{K})$, the set of non-empty closed subsets of the compact set $\mathcal{K} \subset \ell^2$, equipped with the distance $\delta = HD$, the Hausdorff distance in ℓ^2. Take E to be $E := \{I_t^a(M) | M \in \mathcal{M}_{n,k,D}, a \in \mathcal{B}(M)\} \subset \mathcal{F}(\mathcal{K})$. Given $M \in \mathcal{M}_{n,k,D}$, let $I_t(M)$ denote the set $\{I_t^a(M) | a \in \mathcal{B}(M)\} \subset E$. It follows from the proof of Theorem 10 and from the continuity of I (Theorem 8) that $I_t(M)$ is a closed subset of E (w.r.t the distance δ). The set $\mathcal{G}(E) = \{I_t(M) \mid M \in \mathcal{M}_{n,k,D}\}$ is contained in the set $\mathcal{F}(E)$ of non-empty closed subsets of E, equipped with the Hausdorff distance h_δ and hence, by the lemma, $\mathcal{G}(E)$ is precompact for that distance. It is now clear that $h_\delta = d_t$. This finishes the proof of Theorem 14. □

VI. Comparison between Spectral Distances and Lipschitz Distances; C^0-Approximation of Eigenfunctions

In this section we first examine how the spectral distances behave with respect to convergence of metrics, in a very simple case (a product collapsing onto one factor): see Proposition 16. We then investigate the relationships between the spectral distances and the Lipschitz distance. Let g be a given metric on a closed manifold M, with *simple spectrum*. Any sequence $\{g_m\}$ of metrics on M which converges to g in the Lipschitz sense also converges to g for the spectral distance d_t (for any fixed $t > 0$). We also show that this assertion is no longer true, but still partially true, when the spectrum of g has multiplicities: see Theorem 17. For the proof we need the following result which is worth noticing for its own sake: Lipschitz close metrics have C^0-close corresponding eigenfunctions (see Theorem 21 for a precise statement).

PROPOSITION 16. *Let (M^m, g_M) and (N^n, g_N) be two fixed closed Riemannian manifolds. Let $P_p, p \in N^\bullet$, denote the Riemannian product $(M \times N, g_p)$ where $g_p = g_M + {}^1\!/_{p^2} g_N$, which collapses onto (M, g_M) when p goes to infinity. Then, for any fixed $t > 0$, $d_t(P_p, M)$ goes to zero when p goes to infinity.*

Proof: Let $\{\lambda_j\}_{j=0}^{\infty}$ (resp. $\{\mu_\ell\}_{k=0}^{\infty}$) be the eigenvalues of (M, g_M) (resp. (N, g_N)). The eigenvalues ν_ℓ of P_p are of the form $\lambda_j + p^2 \mu_k$. As soon as $p^2 > \lambda_A / \mu_1$, the manifolds M and P_p have the same A first eigenvalues $\lambda_1, \ldots, \lambda_A$. Given any orthonormal basis $\{\varphi_j\}$ (resp. $\{\psi_k\}$), of eigenfunctions of Δ_M (resp. Δ_N), and any orthonormal basis $\{\theta_\ell\}$ of eigenfunctions of the Laplacian on P_1, the function $\theta_\ell(x, y)$ is a linear combination of corresponding functions $\varphi_j(x)\psi_k(y)$ and $\{p^{n/2}\theta_\ell\}$ is an orthonormal basis of eigenfunctions for the Laplacian on P_p. With obvious notations

$$\begin{aligned}
&\left\| I_t^{\theta,p}(x,y) - I_t^{\varphi}(z) \right\|_{\ell^2}^2 \\
&\quad = \sum_{\ell} \left(\sqrt{\operatorname{Vol}(M) \operatorname{Vol}(N)} e^{-\nu_\ell t/2}\, \theta_\ell(x,y) - \sqrt{\operatorname{Vol}(M)} e^{-\lambda_\ell t/2} \varphi_\ell(z) \right)^2 \\
&\quad \leq I + 2\{II + III + IV + V\} ,
\end{aligned}$$

where

$$I = \sum_{\ell \leq A} \left(\sqrt{\operatorname{Vol}(M) \operatorname{Vol}(N)} e^{-\nu_\ell t/2} \theta_\ell(x,y) - \sqrt{\operatorname{Vol} M} e^{-\lambda_\ell t/2} \varphi_\ell(z) \right)^2 ,$$

$$II = \sum_{\substack{\nu_\ell = \lambda_j + p^2 \mu_k \\ j \geq 1, k \geq 1}} \operatorname{Vol}(M) \operatorname{Vol}(N) e^{-\nu_\ell t} \theta_\ell^2(x,y),$$

$$III = \sum_{\substack{\nu_\ell = \lambda_j \\ j \geq A+1}} \operatorname{Vol}(M) \operatorname{Vol}(N) e^{-\nu_\ell t} \theta_\ell^2(x,y),$$

$$IV = \sum_{\substack{\nu_\ell = p^2 \mu_k \\ k \geq 1}} \operatorname{Vol}(M) \operatorname{Vol}(N) e^{-\nu_\ell t} \theta_\ell^2(x,y),$$

$$V = \sum_{j \geq A+1} \mathrm{Vol}(M) e^{-\lambda_j t} \varphi_j^2(z) \,.$$

The following expressions hold:

$$II = \left[\mathrm{Vol}(M) k_M(t,x,x) - 1\right]\left[\mathrm{Vol}(N) k_N(p^2 t, y, y) - 1\right]$$

$$III = \sum_{j \geq A+1} \mathrm{Vol}(M) e^{-\lambda_j t} \varphi_j^2(x)$$

$$IV = \left[\mathrm{Vol}(N) k_N(p^2 t, y, y) - 1\right] \,.$$

As A goes to infinity with p, $II + III + IV + V$ goes to zero uniformly with respect to x, y and the choices of orthonormal bases of eigenfunctions (use Theorem 3). In order to prove that $d_t(P_p, M)$ goes to zero it suffices to estimate the contribution of I. Because of the definition of Hausdorff distances it suffices to make $z = x$ and to take the supremum in (x, y). As the A first eigenvalues of P_p and M coincide, the corresponding eigenfunctions θ_ℓ must be constant on N. It follows that the contribution of I to the spectral distance is zero. □

We recall that the Lipschitz distance between (isometry classes of) Riemannian metrics g and h on a given closed manifold M is defined as the infimum of the constants C such that

$$e^{-c} \varphi^*(g) \leq h \leq e^c \varphi^*(g)$$

for some diffeomorphism φ of M.

THEOREM 17. *Let (M, g) be a closed Riemannian manifold and let $\{g_m\}$ be a sequence of metrics on M which converges to g for the Lipschitz distance. We assume furthermore that the Ricci curvatures of all the metrics under consideration are bounded from below by $-(n-1)K^2$ for some constant K.*

i) If (M, g) has simple spectrum *(i.e. all eigenvalues of the Laplacian have multiplicity one) then g_m converges to g for the spectral distance d_t (for any fixed t).*

ii) In the general case, one still has $\delta_t(g_m, g)$ goes to zero as m goes to infinity where δ_t is "half the spectral distance" defined by

$$\delta_t(h, g) = \sup_{a \in \mathcal{B}(M,h)} \left\{ \inf_{b \in \mathcal{B}(M,g)} \left\{ HD\big(I_t^a(M), I_t^b(M)\big) \right\} \right\} .$$

Remark 18: We will give later on (Example 28) an example of a closed Riemannian manifold (M, g) with a sequence of Riemannian metrics which converge to g in the Lipschitz sense but does not converge to g for the spectral distance d_t.

The proof of Theorem 15 is decomposed into several propositions of independent interest. Recall that (M, g) is a *fixed closed Riemannian manifold.* Let $\Lambda_1 < \Lambda_2 < \cdots < \Lambda_k < \cdots$ be the *distinct* eigenvalues of Δ_g; let E_k and m_k be the corresponding eigenspaces and multiplicities; let $N_k = m_1 + \cdots + m_k$.

Fix some k_0 and let $N_0 = N_{k_0}$. By the continuity of the eigenvalues with respect to the Lipschitz distance, given by the min-max principle, one can find $\varepsilon_0 > 0$ such that the eigenvalues $\lambda_1(h), \ldots, \lambda_{N_0}(h)$ of any metric h satisfying $(1-\varepsilon)g \leq h \leq (1+\varepsilon)g$ with $\varepsilon < \varepsilon_0$ are all less than Λ_{k_0+1} and contained into pairwise disjoint intervals I_k about the Λ_k's, $k \leq k_0$.

We then have

PROPOSITION 19. *With the above notations, let h be any metric on M such that $(1-\varepsilon)g \leq h \leq (1+\varepsilon)g$, $\varepsilon < \varepsilon_0$. There exist constants $\gamma_{g,i}(\varepsilon)$, $1 \leq i \leq N_0$, which go to zero when ε goes to zero, such that to any orthonormal basis $\{\psi_j\}$ of eigenfunctions of Δ_h one can associate an orthonormal basis $\{\varphi_j\}$ of eigenfunctions of Δ_g satisfying $\|\varphi_i - \psi_i\|_{H^1(g)} \leq \gamma_{g,i}(\varepsilon)$ for $i \leq N_0$, where $\|\cdot\|_{H^1(g)}$ is the norm of the Sobolev space H^1 for the metric g on M.*

Proof: Let F_k, $1 \leq k \leq k_0$ be the sum of eigenspaces of Δ_h corresponding to the eigenvalues $\lambda_j(h)$ contained in the interval I_k about Λ_k. Let π_k denote the orthogonal projection in $L^2(g)$ onto the eigenspace E_k.

LEMMA 20. *For $k \leq k_0$, there exist functions $\alpha_k(\varepsilon)$ and $\beta_k(\varepsilon)$ which go to zero with ε such that*

a) $\|(\pi_k - \mathrm{Id})\psi\|_{H^1(g)} \leq \alpha_k(\varepsilon)\|\psi\|_{L^2(h)}$ *for any* $\psi \in F_k$.

b) For any $\varphi \in \bigoplus_{i=1}^{k} E_i$ *and for any* ψ, $L^2(h)$*-orthogonal to* $\bigoplus_{i=1}^{k} F_i$,

$$\left|\langle\varphi, \psi\rangle_{L^2(g)}\right| \leq \beta_k(\varepsilon)\|\varphi\|_{L^2(g)}\|\psi\|_{L^2(h)} .$$

Proof: This lemma is a reformulation of the "Lemme des petites valeurs propres" of Y. Colin de Verdière ([CdV]); see also the version of G. Courtois ([Co]).

The proof is by induction. Let us denote by $\|\cdot\|_g$, resp. Q_g, the $L^2(g)$-norm, resp. the quadratic form, associated with the Laplacian Δ_g and similarly for the metric h. In the sequel $\mathcal{O}(\varepsilon)$ will be a generic function which goes to zero with ε. By the assumptions on the metrics g and h, there exists $\varepsilon' = \mathcal{O}(\varepsilon)$ such that

$$\begin{cases} (1-\varepsilon')\|\psi\|_g^2 \le \|\psi\|_h^2 \le (1+\varepsilon')\|\psi\|_g^2, & \text{for any } \psi \in C^\infty(M) \\ (1-\varepsilon')Q_g(\psi) \le Q_h(\psi) \le (1+\varepsilon')Q_g(\psi), & \text{for any } \psi \in C^\infty(M) \\ (1-\varepsilon')\lambda_j(g) \le \lambda_j(h) \le (1+\varepsilon')\lambda_j(g), & \text{for any } j \ge 1 \end{cases} \qquad (i)$$

Step $k = 1$. Take $\psi \in F_1$, with $\|\psi\|_h = 1$. Write $\psi = \varphi + \varphi''$ with $\varphi = \pi_1(\psi)$. Then φ and φ'' are orthogonal both in $L^2(g)$ and with respect to Q_g. It follows that

$$(1-\varepsilon')\{Q_g(\varphi)+Q_g(\varphi'')\}=(1-\varepsilon')Q_g(\psi)\le Q_h(\psi)=\lambda_1(h)\le\Lambda_1(1+\varepsilon') . \qquad (ii)$$

On the other hand $Q_g(\varphi'') \ge \Lambda_2\|\varphi''\|_g^2$ and $Q_g(\varphi) = \Lambda_1\|\varphi\|_g^2 \ge \Lambda_1\{(1+\varepsilon')^{-1} - \|\varphi''\|_g^2\}$. Plugging these inequalities into *(ii)* we obtain

$$\|\varphi''\|_g^2 = \mathcal{O}(\varepsilon) , \quad \|\varphi\|_g^2 = 1 + \mathcal{O}(\varepsilon) , \quad Q_g(\varphi) = \Lambda_1 + \mathcal{O}(\varepsilon) . \qquad (iii)$$

From *(ii)* and *(iii)* we deduce that $Q_g(\varphi'') = \mathcal{O}(\varepsilon)$. This proves Assertion *a)* of the lemma in the case $k = 1$. It is worth noticing that in *(iii)* the estimate on $\|\varphi\|_g^2$ depends on $\Lambda_2 - \Lambda_1$ and that so does $\alpha_1(\varepsilon)$.

In order to prove Assertion *b)* we first notice that the L^2-inner products of two functions φ and ψ satisfy $|\langle\varphi,\psi\rangle_g - \langle\varphi,\psi\rangle_h| = \mathcal{O}(\varepsilon)\|\varphi\|_g\|\psi\|_h$ (compare the Riemannian measures). Take ψ'' $L^2(h)$-orthogonal to F_1 and φ in E_1. Assertion *a)* implies that π_1 is a bijection from F_1 onto E_1 (for ε small enough) and hence there exists ψ in F_1 such that $\varphi = \pi_1(\psi)$. Now

$$\begin{aligned} \langle\varphi,\psi''\rangle_g &= \langle\varphi,\psi''\rangle_h + \mathcal{O}(\varepsilon)\|\varphi\|_g\|\psi''\|_h \\ &= \langle\varphi-\psi,\psi''\rangle_h + \mathcal{O}(\varepsilon)\|\varphi\|_g\|\psi''\|_h \\ &= \{\|\varphi-\psi\|_g + \mathcal{O}(\varepsilon)\|\varphi\|_g\}\|\psi''\|_h \\ &= \mathcal{O}(\varepsilon)\|\varphi\|_g\|\psi''\|_h \end{aligned}$$

using *(iii)* and Assertion *a)*.

Step $(k-1) \to k$. Assume that the lemma is proved for $F_1, \dots, F_{k-1}$. Define $E'_k = \bigoplus_{i\le k-1} E_i$ and $E''_k = \bigoplus_{i\ge k+1} E_i$. Take any ψ in F_k such that $\|\psi\|_h = 1$, and decompose it as $\varphi' + \varphi + \varphi''$ with respect to the decomposition $E'_k \oplus E_k \oplus E''_k$ of $L^2(g)$. Using inequalities *i)* as above we can write

$$\Lambda_k\frac{1+\varepsilon'}{1-\varepsilon'} \ge (1-\varepsilon')^{-1}Q_h(\psi) \ge Q_g(\psi) = Q_g(\varphi') + \Lambda_k\|\varphi\|_g^2 + Q_g(\varphi'') . \quad (iv)$$

Using Assertion *b)* in the induction hypothesis we have

$$\|\varphi'\|_g^2 = \langle \varphi', \psi \rangle_g \leq \beta_{k-1}(\varepsilon) \|\varphi'\|_g \ . \tag{v}$$

By the min-max, we have $Q_g(\varphi'') \geq \Lambda_{k+1} \|\varphi''\|_g^2$. These last two inequalities and *iv)* lead to

$$\Lambda_k \mathcal{O}(\varepsilon) \geq (\Lambda_{k+1} - \Lambda_k) \|\varphi''\|_g^2$$

which combined with (v) gives $\|\varphi\|_g^2 = 1 + \mathcal{O}(\varepsilon)$. From *(iv)* we also deduce that $Q_g(\varphi') + Q_g(\varphi'') = \mathcal{O}(\varepsilon)$. This proves Assertion a*)* for F_k and shows as before that the functions $\alpha_k(\varepsilon)$ (and $\beta_k(\varepsilon)$) depend on the metric g through its spectrum and the numbers $\Lambda_{k+1} - \Lambda_k$.

In order to prove Assertion *b)*, take ψ'' orthogonal to $\bigoplus_{i \leq k} F_i$, φ' in E_k' and φ in E_k. By Assumption *b)* in the induction hypothesis, we have $\langle \varphi', \psi'' \rangle_g = \mathcal{O}(\varepsilon) \|\varphi'\|_g \|\psi''\|_h$. By Assertion a*)*, π_k is a bijection from F_k onto E_k and hence there is some ψ in F_k such that $\pi_k(\psi) = \varphi$. We can then write

$$\begin{aligned}
\langle \varphi, \psi'' \rangle_g &= \langle \varphi, \psi'' \rangle_h + \mathcal{O}(\varepsilon) \|\varphi\|_g \|\psi''\|_h \\
&= \langle \varphi - \psi, \psi'' \rangle_h + \mathcal{O}(\varepsilon) \|\varphi\|_g \|\psi''\|_h \\
&= \mathcal{O}(\varepsilon) \|\varphi\|_g \|\psi''\|_h
\end{aligned}$$

using *i)* and Assertion a*)* again. This proves Assertion *b)* for F_k and finishes the proof of Lemma 20.

Proof of Proposition 19: Take any orthonormal basis $\{\psi_i\}$ of eigenfunctions of Δ_h. For each $k \leq k_0$ it defines an orthonormal basis of F_k, say $\psi_1^k, \ldots, \psi_{m_k}^k$. For ε small enough, we can orthonormalize the family $\pi_k(\psi_1^k) \cdots \pi_k(\psi_{m_k}^k)$ of E_k (Gram-Schmidt) to an orthonormal basis $\varphi_1^k, \ldots, \varphi_{m_k}^k$. By Lemma 20 the family $\pi_k(\psi_1^k), \ldots, \pi_k(\psi_{m_k}^k)$ is almost orthonormal and hence $H^1(g)$-close to $\varphi_1^k, \ldots, \varphi_{m_k}^k$. Lemma 20 again shows that $\psi_1^k, \ldots, \psi_{m_k}^k$ and $\varphi_1^k, \ldots, \varphi_{m_k}^k$ are $H^1(g)$-close. This proves Proposition 19. □

To finish the proof of Theorem 17 we need the next result which seems to be of independent interest (in the Theorem, ε_0 is the same as the one defined for Proposition 19 supra).

THEOREM 21. *With the above notations, let h be any metric on M such that $(1 - \varepsilon)g \leq h \leq (1 + \varepsilon)g$, $\varepsilon < \varepsilon_0$. We assume furthermore that the metrics under consideration have their Ricci curvatures bounded from below by $-(n-1)K^2$ for some constant K. There exist constants $\eta_{g,i,K}(\varepsilon)$, $1 \leq i \leq N_0$, which go to zero with ε, such that to any orthonormal basis $\{\psi_j\}$*

of eigenfunctions of Δ_h *one can associate an orthonormal basis* $\{\varphi_i\}$ *of eigenfunctions of* Δ_g *satisfying* $\|\varphi_i - \psi_i\|_\infty \leq \eta_{g,i,K}(\varepsilon)$ *for* $i \leq N_0$, *where* $\|\cdot\|_\infty$ *is the sup-norm.*

Proof of Theorem 21: For $p > \dim(M)$ we have the Sobolev inequality

$$\|u\|_\infty \leq K_p^{(1)}(g)\big(\|du\|_p + \|u\|_p\big)$$

for any $u \in C^\infty(M)$ from which we deduce an inequality of the following form for any $u \in C^\infty(M)$

$$\|u\|_\infty \leq K_p^{(2)}(g)\|u\|_{H^1(g)}^{2/p}\big(\|u\|_\infty + \|du\|_\infty\big)^{(p-2)/p} . \tag{22}$$

Let $\{\psi_i\}$ be an orthonormal basis of eigenfunctions of Δ_h and let $\{\varphi_i\}$ be the basis of eigenfunctions of Δ_g associated with $\{\psi_i\}$ and given by Proposition 19. Because the metric h is ε-Lipschitz close to g, the norms we consider, for either the metric g or the metric h are comparable: the ratios of corresponding norms are bounded between, say $1/2$ and 2. We conclude that for some $K_p^{(3)}(g)$ and any $i \geq 1$,

$$\begin{gathered}\|\varphi_i - \psi_i\|_\infty \leq \\ K_p^{(3)}(g)\|\varphi_i-\psi_i\|_{H^1(g)}^{2/p}\big\{\|\varphi_i\|_\infty+\|\psi_i\|_\infty+\|d\varphi_i\|_{L^\infty(g)}+\|d\psi_i\|_{L^\infty(g)}\big\}\end{gathered} \tag{23}$$

In order to conclude the proof of Theorem 21, it suffices to give uniform bounds for $\|\varphi_i\|_\infty$, $\|\psi_i\|_\infty$, $\|d\varphi_i\|_{L^\infty(g)}$ and $\|d\psi_i\|_{L^\infty(h)}$ (which controls $\|d\psi_i\|_{L^\infty(g)}$) for $i \leq N_0$. Let us consider $\|\psi_i\|_\infty$ and $\|d\psi_i\|_{L^\infty(h)}$; the metrics have their Ricci curvatures bounded below by $-(n-1)K^2$ by assumption, they also have their diameters bounded from above by some $D > 0$ because h is ε-Lipschitz close to the fixed metric g. Their N_0 first eigenvalues are bounded from above by $(1 + O(\varepsilon))\lambda_{N_0}(g)$ (due to the ε-closeness to g). Applying Theorem 3 *(iii)* we can conclude that there exists some $C_1(N_0, K, g, n)$ such that

$$\|\varphi_i\|_\infty, \|\psi_i\|_\infty \leq C_1(N_0, K, g, n) \quad \text{for} \quad i \leq N_0 . \tag{24}$$

Now the 1-form $d\psi_i$ is an eigenform for the Hodge-de Rham Laplacian $\Delta^{(1)}$ acting on 1-forms for the eigenvalue $\lambda_i(h)$. By Kato's inequality for $e^{-\Delta^{(1)}t}$ (see [HSU] and [BG]), and Bochner's formula

$$\mathrm{Trace}(e^{-t\Delta^{(1)}}) = \mathrm{Trace}(e^{-t(\bar{\Delta}+\mathrm{Ricci})}) \leq n\,\mathrm{Trace}(e^{-t(\Delta-(n-1)K^2)})$$

which implies

$$e^{-\lambda_i(h)t}\|d\psi_i\|_h^{-2}\left|d\psi_i(x)\right|_h^2 \leq ne^{t(n-1)K^2}k_h(t;x,x)$$

where k_h is the (scalar) heat kernel for the metric h. Applying Theorem 3 *(iii)* again, we may conclude that there exists a constant $C_2(N_0, K, g, n)$ such that

$$\|d\psi_i\|_\infty \leq C_2(N_0, K, g, n) \text{ for } i \leq N_0 \tag{25}$$

and similarly for $\|d\varphi_i\|_\infty$.

It now suffices to apply Proposition 19. □

End of the proof of Theorem 17: Let g and h be ε-Lipschitz close as in Theorem 17. Let a and b be elements of $\mathcal{B}(M,g)$ and $\mathcal{B}(M,h)$. As in Lemma 20, there exists ε' (going to zero with ε) such that, for any j

$$(1-\varepsilon')\lambda_j(g) \leq \lambda_j(h) \leq (1+\varepsilon')\lambda_j(g)\ . \tag{26}$$

By the definition of I_t^a and I_t^b, we have

$$\left\{\begin{aligned} HD^2\left(I_t^a(M), I_t^b(M)\right) \leq 2 &\sum_{1\leq i\leq k} e^{-\lambda_j t}\|\widetilde{\varphi}_j - \widetilde{\psi}_j\|_{L^\infty}^2 + \cdots \\ &+ 2\sup_x\left\{\sum_{1\leq j\leq k}(e^{\lambda_j t\varepsilon'/2} - 1)^2 e^{-\lambda_j t}\widetilde{\psi}_j^2(x)\right\} + \cdots \\ &+ 2\sup_x\left\{\sum_{k+1\leq j} e^{-\lambda_j t}\widetilde{\varphi}_j^2(x)\right\} + 2\sup_x\left\{\sum_{k+1\leq j} e^{-\mu_j t}\widetilde{\psi}_j^2(x)\right\} \end{aligned}\right. \tag{27}$$

where $\lambda_j, \mu_j, \widetilde{\varphi}_j, \widetilde{\psi}_j$ stand for $\lambda_j(g)$, $\lambda_j(h)$, $\sqrt{\mathrm{Vol}(M,g)}\varphi_j$, and $\sqrt{\mathrm{Vol}(M,h)}\psi_j$. By the same proof as the one given for Theorem 3 (iii), the two last terms of (27) are bounded from above by

$$E(n,K,D)t^{-n/2}\int_{t\lambda_{k+1}}^{+\infty} s^{n/2}e^{-s}\, ds\ .$$

Let $I(u) = \int_u^{+\infty} s^{n/2}e^{-s}\, ds$. The function $u\, I(u)$ goes to zero when u goes to zero or $+\infty$, hence it has a maximum, denoted I_0. Given any $\eta > 0$, let us choose k such that $\lambda_k \leq {}^{I_0 E}\!/_{\eta} \leq \lambda_{k+1}$, it follows that the two last terms of (27) are both bounded by $\eta t^{-(n+2)/2}$. It also follows that the second term of the right-hand side of (27) is bounded by $(e^{tI_0E\varepsilon'/2\eta} - 1)^2\sum_{j=1}^{+\infty} e^{-\lambda_j t}\widetilde{\psi}_j^2(x)$; by Theorem 3 *(iii)* this term is bounded from above by

$$C(n,K,D)t^{-n/2}(e^{tI_0E\varepsilon'/2\eta} - 1)^2 \leq \eta t^{-(n+2)/2}$$

for ε' small enough (precisely $\varepsilon' \le \mathrm{C}^{\mathrm{te}}(\eta/t)^{3/2}$).

Plugging these inequalities into (27) gives

$$HD^2\big(I_t^a(M), I_t^b(M)\big) \le 2 \sum_{\lambda_i \le I_0 E'/\eta} e^{-\lambda_j t}\|\widetilde{\varphi}_j - \widetilde{\psi}_j\|^2_{L^\infty} + 6\eta t^{-(n+2)/2} .$$

From Theorem 21, for any choice of $b \in \mathcal{B}(M,h)$, there exists $a \in \mathcal{B}(M,g)$ such that

$$\sum_{\lambda_i \le I_0 E/\eta} e^{-\lambda_j t}\|\widetilde{\varphi}_j - \widetilde{\psi}_j\|^2_{L^\infty} < \eta t^{-(n+2)/2}$$

and this proves that $\delta_t\big[(M,g),(M,h)\big] < 8\eta t^{-(n+2)/2}$ (we recall that t is fixed).

Conversely, if each eigenvalue $\lambda_1(g), \dots, \lambda_k(g)$ has multiplicity 1, there exists a choice of ε (and then an ε') such that $\lambda_{i+1}(h) - \lambda_i(h)$ is bounded from below for any $i \in \{1, \dots, k-1\}$. As K is assumed to be fixed, all the invariants occurring in the estimation of $\gamma_{h,i}(\varepsilon)$ and $\eta_{h,i,K}(\varepsilon)$ do not depend on the particular choice of h, as long as h is assumed to be ε-Lipschitz close to g. This shows that

$$d_t\big((M,g),(M,h)\big) < \eta . \qquad \square$$

EXAMPLE 28: We now proceed to the description of the example announced in the beginning of this section.

Let us consider the 2-torus and the metric $g_\varepsilon = d\theta^2 + (1+\varepsilon)^2 d\varphi^2$. Let $\{\Phi_1, \Phi_2, \Phi_3, \Phi_4\}$ be the basis of the first eigenspace of Δ_{g_0}, defined by

$$\begin{aligned} \Phi_1(\theta,\varphi) &= \frac{1}{2\pi}(\cos\theta + \cos\varphi) , & \Phi_2(\theta,\varphi) &= \frac{1}{2\pi}(\cos\theta - \cos\varphi) \\ \Phi_3(\theta,\varphi) &= \frac{1}{2\pi}(\sin\theta + \sin\varphi) , & \Phi_4(\theta,\varphi) &= \frac{1}{2\pi}(\sin\theta - \sin\varphi) \end{aligned}$$

Any basis $\{\Psi_1^\alpha, \Psi_2^\alpha\}$ and $\{\Psi_3^\beta, \Psi_4^\beta\}$ of the two first eigenspaces of Δ_{g_ε} can be written

$$\begin{aligned} \Psi_1^\alpha(\theta,\varphi) &= \frac{1}{\pi\sqrt{2(1+\varepsilon)}}\cos(\varphi+\alpha), & \Psi_2^\alpha(\theta,\varphi) &= \pm\frac{1}{\pi\sqrt{2(1+\varepsilon)}}\sin(\varphi+\alpha) \\ \Psi_3^\beta(\theta,\varphi) &= \frac{1}{\pi\sqrt{2(1+\varepsilon)}}\cos(\theta+\beta), & \Psi_4^\beta(\theta,\varphi) &= \pm\frac{1}{\pi\sqrt{2(1+\varepsilon)}}\sin(\theta+\beta) . \end{aligned}$$

For any choice of (α,β) one has

$$\inf_{(\theta,\varphi)} \big|\Phi_1(0,0) - \Psi_1^\alpha(\theta,\varphi)\big| \ge \frac{2-\sqrt{2}}{2\pi}$$

and hence if one extends $\{\Phi_1, \dots, \Phi_4\}$ (resp. $\{\Psi_1^\alpha, \Psi_2^\alpha, \Psi_3^\beta, \Psi_4^\beta\}$) to an element $a \in \mathcal{B}(M, g_0)$ (resp. $a' \in \mathcal{B}(M, g_\varepsilon)$), one has

$$\sup_{a \in \mathcal{B}(M,g_0)} \inf_{a' \in \mathcal{B}(M,g_\varepsilon)} HD(I_t^a(M), I_t^{a'}(M')) \geq C_n(t) > 0$$

showing that $d_t(g_0, g_\varepsilon)$ does not go to zero when ε goes to zero, while $\delta_t(g_\varepsilon, g_0)$ does.

Clearly this example is typical and the fact that $\delta_t(g_0, g_\varepsilon)$ does not go to zero comes from the fact that $\bigcup_{a \in \mathcal{B}(M,g_0)} I_t^a(M)$ is much bigger (because of the multiplicities) than $\bigcup_{a' \in \mathcal{B}(M,g_\varepsilon)} I_t^{a'}(M)$. □

VII. Continuity of Eigenvalues w.r.t. Spectral Distances

THEOREM 29. *The eigenvalues of the Laplacian are continuous with respect to the spectral distances.*

Proof: Let M_k be a sequence of manifolds which converges to M for the spectral distance d_t. In particular, for any $a \in \mathcal{B}(M)$, there exists some $b_k \in \mathcal{B}(M_k)$ such that

$$HD_{\ell^2}\left(I_t^{b_k}(M_k), I_t^a(M)\right) \leq \varepsilon_k < \varepsilon_0 \qquad (a)$$

where ε_0 is the size of a fixed tubular neighborhood of the closed submanifold $I_t^a(M) \subset \ell^2$. To such a tubular neighborhood is associated a continuous projection map $\pi : \mathrm{Tub}_{\varepsilon_0}(I_t^a(M)) \to I_t^a(M)$. Introduce the continuous maps $\pi_k = (I_t^a)^{-1} \circ \pi \circ I_t^{b_k} : M_k \to M$ and the image probability measures $\mu_k = (\pi_k)_* dv_{M_k} / \operatorname{Vol}(M_k)$.

Condition (a) on the Hausdorff distance implies that for all $x \in M_k$,

$$\sum_{i=1}^{\infty} \left(\sqrt{\operatorname{Vol}(M_k)} e^{-\lambda_i(M_k)t/2} \psi_i^{b_k}(x) - \sqrt{\operatorname{Vol}(M)} e^{-\lambda_i(M)t/2} \varphi_i^a(\pi_k(x))\right)^2 \leq \varepsilon_k^2.$$

Integrating the inequalities

$$-\varepsilon_k \leq \sqrt{\operatorname{Vol}(M_k)} e^{-\lambda_i(M_k)t/2} \psi_i^{b_k}(x) - \sqrt{\operatorname{Vol}(M)} e^{-\lambda_i(M)t/2} \varphi_i^a\left(\pi_k(x)\right) \leq \varepsilon_k \qquad (b)$$

on M_k, with respect to the probability measure $dv_{M_k} / \operatorname{Vol}(M_k)$, gives

$$\forall i, \quad \left| \sqrt{\operatorname{Vol}(M)} e^{-\lambda_i(M)t/2} \int_M \varphi_i^a(y) d\mu_k(y) \right| \leq \varepsilon_k \ . \qquad (c)$$

Take any converging subsequence of the sequence $\{\mu_k\}$. The limit is a probability measure μ on M. By (c), this measure satisfies $\mu(\varphi_i) = 0$ for all $i \geq 1$ and it follows that $\mu = dv_M/\mathrm{Vol}(M)$ and that the sequence $\{\mu_k\}$ itself converges. Integrating inequality (b) squared (with respect to $dv_{M_k}/\mathrm{Vol}(M_k)$), we can conclude that for any fixed i, and any fixed t, $e^{-\lambda_i(M_k)t}$ tends to $e^{-\lambda_i(M)t}$ as k tends to infinity. □

VIII. Further Comments

a) Embedding into an infinite dimensional sphere.

In order to "renormalize" the embedding ψ_t^a and avoid the collapsing on a point when t goes to zero, we can use the following family of maps

$$M \longrightarrow \ell^2$$

$$K_t^a : x \longrightarrow \frac{1}{\left(\sum\limits_{j\geq 1} e^{-\lambda_j t}\varphi_j^2(x)\right)^{1/2}}\{e^{-\lambda_j t/2}\varphi_j(x)\}_{j\geq 1} \ .$$

PROPOSITION 30. *The map K_t^a is an embedding for each $t > 0$; furthermore*

i) $K_t^a(M) \subset S^\infty$ the unit sphere of ℓ^2.

ii) if g_t' is the metric $g_t' = (K_t^a)^*\,\mathrm{can}$, then $g_t' = 1/2t[g - t/3\,\mathrm{Ricci} + O(t^2)]$ when t goes to zero.

Remark: Any minimal submanifold of a canonical sphere is embedded by an eigenspace [L]. Is K_t asymptotically minimal in any reasonable sense?

b) Embedding the unitary bundle.

The precompactness theorem (Theorem 8) relies on Theorem 3, *i.e.* relies on the estimate of $k_M(t;x,x)$ and the corresponding one on $Z_M(t)$ (or equivalently $\mathrm{Vol}(M)k_M(t;x,x) - 1$ and $Z_M(t) - 1$).

Any such estimate would yield a precompactness result. For example, let us call UM the total space of the unitary bundle of M endowed with the canonical Riemannian metric given by M, it was proved in [BG] and [Bes1] that

PROPOSITION 31. *With the above notations*

$$0 \leq k_{UM}(t;u,u) \leq k_M(t;x,x)k_{S^{n-1}}(t;p,p) \ .$$

For u a point in UM which projects down onto x and p any point in the standard sphere S^{n-1}. Consequently

$$Z_{UM}(t) \leq Z_M(t)Z_{S^{n-1}}(t) \ .$$

This enables us to give the following improvement of Theorem 8. Let us embed $M \in \mathcal{M}_{n,k,D}$ into ℓ^2 by the "heat kernel" of UM. Then by the same technique as in section IV we can define a distance on $\mathcal{M}_{n,k,D}$, say D_t, using the Hausdorff distance on subset of ℓ^2. We then have

PROPOSITION 32. *For any fixed t the metric space $(\mathcal{M}_{n,k,D}, D_t)$ is precompact.*

The proof is easy, it uses Proposition 31 above and the estimates of Theorem 1.

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