Partition function and Heisenberg group

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Abstract

The partition function is a classical function that has been introduced in the mid 1800s. This function can be seen as a potential on the Heisenberg group. This new point of view is very helpful to understand the positive harmonic functions on this group.

The partition function $p(x,y,z)$ is defined, for integers x,y,z, as the number of ways to write z as a sum of integers

$$z = n_1 + \cdots + n_y \quad \text{with} \quad x \geq n_1 \geq \cdots \geq n_y \geq 0\,.$$

Such a decomposition is called a "partition" of z. By convention, one sets $p(x,y,z) = 0$ when x, y or z is negative. This partition function is non-zero for $x \geq 0$, $y \geq 0$, $xy \geq z \geq 0$, and satisfies the equalities

$$p(x,y,z) = p(y,x,z) = p(x,y,xy-z).$$

The proof of these equalities is a nice exercise. This function has been studied for almost two hundred years. For instance, Cayley and Sylvester proved in the 1850s that the sequence

$z \mapsto p(x,y,z)$ *is increasing for* $z \leq xy/2$ *and decreasing for* $z \geq xy/2$.

This fact looks very elementary, but two hundred years later, it still does not have a purely combinatorial proof.

The partition function also satisfies the functional equation,

$$p(x,y,z) = p(x-1,y,z-y) + p(x,y-1,z), \tag{0.1}$$

for all $(x,y,z) \neq 0$. One checks it by splitting the set of partitions according to the colour of the lower-left case of the rectangle as in Figure 1.

The Heisenberg group $G = H_3(\mathbb{Z})$ is the set of triples of integers $g = (x,y,z)$ endowed with the product

$$(x_0,y_0,z_0)\,(x,y,z) = (x_0+x, y_0+y, z_0+z+x_0y)\,.$$

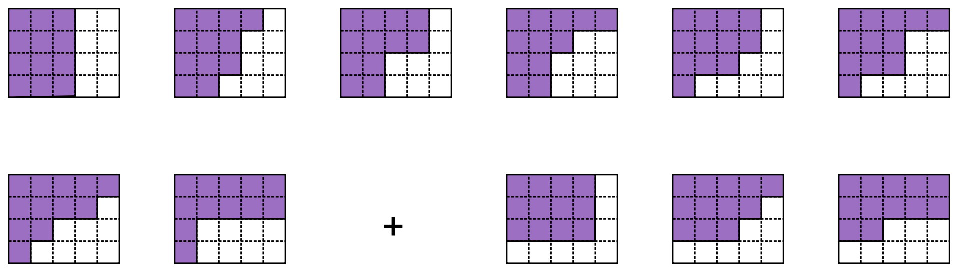

Figure 1: The equality $p(5,4,12) = p(4,4,8) + p(5,3,12)$, that is $11 = 8 + 3$.

Let $S \subset G$ be a weighted finite set. This means that each element s of S has a mass $\mu_s > 0$. For a function $h(x, y, z)$ on G we define its "weighted sum of left translates"

$$Ph(g) = \sum_{s \in S} \mu_s \, h(s^{-1}g).$$

Choosing $S = \{a, b\}$ with $a = (1, 0, 0)$ and $b = (0, 1, 0)$ having weight 1. The functional equation (0.1) says that, outside point 0, the function $h = p$ satisfies $h = Ph$.

The potential at 0 of this *Markov operator* P is nothing but this partition function p. That is, one has the equality,

$$p = \sum_{n=0}^{\infty} P^n \mathbf{1}_{\{0\}} \, ,$$

where $\mathbf{1}_{\{0\}}$ is the characteristic function at 0. Indeed, as can be seen in Figure 2, for g in G, $p(g)$ is the number of ways to write g as a word in a and b.

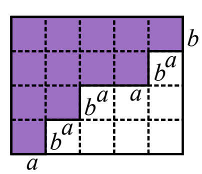

Figure 2: The partition 12=5+4+2+1 associated to the word *ababaabab* gives the element $g = ababaabab = (5, 4, 12)$ in G.

The P-harmonic functions on G are the functions h on G that satisfy $h = Ph$, that is, for our example $S = \{a, b\}$,

$$h(x, y, z) = h(x-1, y, z-y) + h(x, y-1, z). \tag{0.2}$$

We want to describe the positive P-harmonic functions[1] on G, that is, the positive solutions of this infinite family of linear equations (0.2). By a very general theorem of Choquet, it is enough to describe the extremal positive P-harmonic functions on G that is those that cannot be written as the sum of two non-proportional positive P-harmonic functions.

When h does not depend on z, one can choose h to be a character,

$$h(x,y,z) = r^x s^y \quad \text{with } r,s>0 \text{ and } \quad 1/r + 1/s = 1.$$

When h does not depend on x one can choose h to be the partition function in a very wide rectangle,

$$h(x,y,z) = p_\infty(y,z) := p(z,y,z).$$

	↑ z									
0	**0**	1	5	10	15	18	20	21	22	
0	**0**	1	4	8	11	13	14	15	15	
0	**0**	1	4	7	9	10	11	11	11	
0	**0**	1	3	5	6	7	7	7	7	
0	**0**	1	3	4	5	5	5	5	5	
0	**0**	1	2	3	3	3	3	3	3	
0	**0**	1	2	2	2	2	2	2	2	
0	**0**	1	1	1	1	1	1	1	1	
0	1	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	→ y
0	**0**	0	0	0	0	0	0	0	0	

Figure 3: The function p_∞ satisfies $p_\infty(y,z) = p_\infty(y, z-y) + p_\infty(y-1, z)$. The red diagonal is the partition function studied by Hardy and Ramanujan.

The following theorem describes all the P-harmonic functions and it can be extended to any weighted finite subset S of the Heisenberg group G.

Theorem. *An extremal positive P-harmonic function on G is*
either a character $h(x,y,z) = r^x s^y$,
or a partition function $h(x,y,z) := p_\infty(y,z)$,
or a switch of the partition function $h(x,y,z) := p_\infty(x, xy-z)$,
or a right translate of a multiple of one of these functions.

[1]The positive P-harmonic functions where described by Choquet and Deny on abelian groups in the 1950s, by Margulis on nilpotent groups when S spans G as a semigroup in the 1960s, and by Ancona on hyperbolic groups in the 1980s.